WAVE TRANSFORMATION IN A MEDIUM WITH RANDOM INHOMOGENEITIES

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The effect of wave transformation in a weakly irregular medium consists in the following. Suppose that two kinds of coupled oscillation are possible, h_1 and h_2, described by the equations

$$\frac{d^2h_1}{dx^2} + k_1^2(x)h_1 = \alpha(x)h_2,$$

$$\frac{d^2h_2}{dx^2} + k_2^2(x)h_2 = \alpha(x)h_1. \quad (1)$$

Here x is the irregularity parameter. In a uniform medium we may pass to normal oscillations $H_{1,2}$:

$$\frac{d^2H_{1,2}}{dx^2} + q_{1,2}^2 H_{1,2} = 0,$$

where the wave vectors $q_{1,2}$ of the normal oscillations are determined from the equations

$$q_{1,2}^2 = {}^1/_2(k_1^2 + k_2^2) \pm \sqrt{{}^1/_4(k_1^2 - k_2^2)^2 + \alpha^2}.$$

In a weakly irregular medium, when $k_{1,2}$ and α are "slowly varying" functions of the coordinates

$$\frac{d}{dx}(\ln q_{1,2}) \ll q_{1,2}; \quad (3)$$

the "quasi-normal" oscillations

$$H_{1,2}^{\pm} \approx \frac{1}{\sqrt{q_{1,2}}} \exp\left\{\pm i \int^{x} q_{1,2}(x')\,dx'\right\} \quad (4)$$

are approximate solutions of (1), where $q_{1,2}$ are determined from Eq. (2) as before.

In certain areas, and, in particular, in the neighborhood on the points where $q_1 = q_2$, solutions of the type (4) become invalid. In passing through resonance regions of this type the amplitudes of the quasi-normal oscillations suffer a discontinuous change compared with their initial values (the Stokes' phenomenon) and there is a redistribution of energy between the quasi-normal modes of oscillation. The term "wave transformation" will be used to describe just this phenomenon, and the resonance points where $q_1 = q_2$ will be called transformation points. The phenomenon of wave transformation in a weakly nonhomogeneous medium has been fairly well studied in connection with various problems in astrophysics [1–3] and plasma heating [4–6]. As a formal basis for calculating the transformation coefficients we may use the method developed by Stueckelberg [7] for the system of equations (1), consisting of matching the asymptotic solutions (4) in passing through the neighborhood of a transformation point.

When the wave traverses a sufficiently large volume of plasma, the number of transformation points may be large. Their distribution may naturally be taken to be random and given in the form of some random function of the coordinate. The question arises of describing the kinetics of waves in a medium with randomly placed transformation points. The problem bears a formal similarity to a system of coupled oscillators passing through resonances at random moments in time. A method of solving problems of this type is developed below.

We shall commence by considering a single transformation event. Let the solution of Eq. (1) be represented in the form

$$H = A_1H_1^+ + A_2H_2^+,$$

for some values of x to the left of the transformation region.

On the right of the transformation region the solution has the form

$$H = A_1^*H_1^+ + A_2^*H_2^+.$$

Here the relation between (A_1^*, A_2^*) and (A_1, A_2) is determined by the equation [8]

$$\begin{pmatrix} A_1^* \\ A_2^* \end{pmatrix} = \begin{pmatrix} ie^{i\varphi}\cos a & \sin a \\ \sin a & ie^{-i\varphi}\cos a \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix},$$

$$\sin a = e^{-\delta}, \quad \delta = \frac{1}{2}\left|\oint (q_1 - q_2)\,dx\right|. \quad (5)$$

Here the integral in δ is taken along a contour enclosing two complex conjugate transformation points; φ is the phase which is known and is not important in what follows. Each transformation event may be regarded as a wave "collision," and the transition matrix from (A_1, A_2) to (A_1^*, A_2^*) as the collision operator.

The transition matrix for successive collisions has the form

$$M = \begin{pmatrix} ie^{i\varphi+iS_1}\cos a & e^{iS_2}\sin a \\ e^{iS_1}\sin a & ie^{-i\varphi+iS_2}\cos a \end{pmatrix},$$

$$S_1 = \int q_1(x')\,dx', \quad S_2 = \int q_2(x')\,dx'. \quad (6)$$

Here the integrals in $S_{1,2}$ are taken between the two closest transformation points. In order to avoid the possibility of transformation regions overlapping, we confine ourselves to the case of comparatively infrequent collisions and require that

$$lq_{1,2} \gg 1, \quad (7)$$

where l is the mean distance between transformation points. Inequality (7) leads, in particular, to the fact that the phase advances S_1 and S_2 in (6) are large and so the phase φ may be neglected.

We shall now assume that the vector A_0 with components $(A_1^{(0)}, A_2^{(0)})$ is given at some initial point x_0, and in the path segment to x the wave experiences n collisions (passes through n transformation points). Then the vector A_n may be represented in the following form at points x:

$$A_n(x) = M_n M_{n-1} \cdots M_1 A_0(x_0).$$

Here $M_k = M_k(x_{k-1}, x_k)$ and is determined from formula (6), where x_k is a transformation point, all parameters depend on the number k, and the integrals in

$S^{(k)}_{1,2}$ are calculated on the arc between x_{k-1} and x_k. The problem consists in determining the mean values of $\langle A(x) \rangle$ averaged over all possible configurations of transformation point dispositions on (x_0, x). We shall take the transformation points to be distributed according to a Poisson distribution, and the quantity a to be constant for the moment (the restriction on a will be removed later). This means that the probability of a transformation point occuring in an element dx is l^{-1} dx.

We shall consider the system

$$\frac{dU}{dx} = iq_1U - i\sum_k \delta(x - x_k)\left(aV - \frac{\pi}{2}U\right)$$
$$\frac{dV}{dx} = iq_2V - i\sum_k \delta(x - x_k)\left(aU - \frac{\pi}{2}V\right), \qquad (8)$$

where x_k are transformation points. It is not difficult to establish that the transition matrix of solutions of system (8) between two successive transformation points is identical with (6) if we set

$$U = \sqrt{q_1}H_1, \qquad V = \sqrt{q_2}H_2. \qquad (9)$$

It follows from (9) that the square of the amplitudes U, V coincide with the effects of the H_1- and H_2-oscillations, respectively, and the problem of averaging the solutions of system (1) may be replaced by the equivalent problem of averaging the solutions of system (8).

We now introduce the distribution function $f(x, U_1, U_2, V_1, V_2)$, where

$$U_1 = \mathrm{Re}U, \qquad U_2 = \mathrm{Im}\, U, \qquad V_1 = \mathrm{Re}V,$$
$$V_2 = \mathrm{Im}\; V \qquad \int f\, dU_1 dU_2 dV_1 dV_2 = 1.$$

The kinetic equation for f may be obtained in the usual manner (see, for example, [9])

$$\frac{\partial f}{\partial x} - q_1U_2\frac{\partial f}{\partial U_1} + q_1U_1\frac{\partial f}{\partial U_2} -$$
$$- q_2V_2\frac{\partial f}{\partial V_1} + q_2V_1\frac{\partial f}{\partial V_2} = S^*\{f\} \qquad (10)$$

where the collision term has the form

$$S^*\{f\} = \frac{1}{l}\,[f(x, U_1^*, U_2^*, V_1^*, V_2^*) - f],$$
$$f = f(x, U_1, U_2, V_1, V_2). \qquad (11)$$

The coordinates $U_{1,2}^*$, $V_{1,2}^*$ are determined from the condition that they take the values $U_{1,2}$, $V_{1,2}$ as a result of a collision. Equations (10), (11) have the form of an ordinary Kolmogorov-Feller equation for a discontinuous random process. From system (8), or from (5) and (9), we have

$$U_1^* = U_2\cos a + V_1 \sin a,$$
$$U_2^* = -U_1\cos a + V_2\sin a,$$
$$V_1^* = U_1 \sin a + V_2\cos a,$$
$$V_2^* = U_2 \sin a - V_1\cos a. \qquad (12)$$

The quantity

$$I = |U|^2 + |V|^2 = q_1|H_1|^2 + q_2|H_2|^2 \qquad (13)$$

is invariant under the transformation of (12) as well as of (5) and (6), and is the total effect of a system of two oscillations. The effect of the collisions consists of redistributing the adiabatic invariants of each oscillation.

Equations (10), (11) allow us to calculate any moment of the distribution function f. It is of physical interest to calculate the average values of the adiabatic invariants for each type of oscillation, i.e., in accordance with (13), the averages $\langle |U|^2 \rangle$, $\langle |V|^2 \rangle$. Multiplying (10) in turn by U_1^2, U_2^2, U_1U_2, V_1^2... and integrating over the entire phase space we obtain

$$\frac{d\langle I_1\rangle}{dx} = -\frac{\sin^2 a}{l}\langle I_1\rangle + \frac{\sin^2 a}{l}\langle I_2\rangle + \frac{\sin 2a}{l}\langle I_{21}\rangle,$$
$$\frac{d\langle I_2\rangle}{dx} = \frac{\sin^2 a}{l}\langle I_1\rangle - \frac{\sin^2 a}{l}\langle I_2\rangle - \frac{\sin 2a}{l}\langle I_{21}\rangle,$$
$$\frac{d\langle I_{12}\rangle}{dx} = (q_2 - q_1)\langle I_{21}\rangle,$$
$$\frac{d\langle I_{21}\rangle}{dx} = -(q_2 - q_1)\langle I_{12}\rangle - \frac{\sin 2a}{2l}\langle I_1\rangle +$$
$$+ \frac{\sin 2a}{2l}\langle I_2\rangle - 2\frac{\sin^2 a}{l}\langle I_{21}\rangle, \qquad I_1 = U_1^2 + U_2^2,$$
$$I_2 = V_1^2 + V_2^2, \quad I_{12} = U_1V_1 + U_2V_2 = \mathrm{Re}\, U\overline{V},$$
$$I_{21} = U_1V_2 - U_2V_1 = -\,\mathrm{Im}U\overline{V}. \qquad (14)$$

We may find the steady-state solution from (14) and (13):

$$\langle I_1\rangle = \langle I_2\rangle = {}^1/_2 I, \qquad \langle I_{12}\rangle = \langle I_{21}\rangle = 0. \qquad (15)$$

The result (15) means in particular that if an oscillation with a given value of I only is excited at the plasma boundary, then on passing through a sufficiently wide layer the second oscillation is aroused to a considerable extent.

We now describe the process of approaching equilibrium. We look for a solution of system (14) in the form $\sim e^{\varkappa x}$. The equation for $\varkappa$ is

$$\varkappa^3 + 4\frac{\sin^2 a}{l}\varkappa^2 + \left[4\frac{\sin^2 a}{l^2} + (q_1 - q_2)^2\right]\varkappa +$$
$$+ 2(q_1 - q_2)^2\frac{\sin^2 a}{l} = 0. \qquad (16)$$

Of the three roots of Eq. (16) one is negative and two are complex conjugates with negative real parts.

The relaxation length is determined by the root $\varkappa_0$ for which $|\mathrm{Re}\,\varkappa_0|$ is a minimum. We shall write out the values of $\varkappa_0$ for some limiting cases:

$$\varkappa_0 \approx -{}^1/l_0, \qquad l_0|q_1 - q_2| \gg 1, \qquad l_0 = l/4\sin^2 a,$$
$$\varkappa_0 \approx -{}^1/_2(q_1 - q_2)^2 l_0; \qquad l_0|q_1 - q_2| \ll 1. \qquad (17)$$

In view of the condition that collisions be infrequent (7), the second case can occur only for sufficiently small values of $(q_1 - q_2)$.

Now if the collision parameter a is taken to be random with a distribution function $w(a)$,

$$\int w(a)\, da = 1,$$

then the collision term $S^*\{f\}$ in Eq. (10) is replaced by

$$\langle\langle S^*\{f\}\rangle\rangle = \int w(a) S^*\{f(a)\}\, da.$$

Similarly $\sin^2 a$ in Eq. (16) must be replaced by

$$\langle\langle \sin^2 a \rangle\rangle = \int w(a) \sin^2 a \, da \, .$$

In conclusion we make two observations. The first is connected with the fact that we have treated only the transformation points $q_1 = q_2$. However, there exist other singular points in the solutions of (4), for example at points where $q_{1,2} = 0$. It has been shown in [10] that points of this type lead to a general increase in the mean of the adiabatic invariant I of the whole system. The treatment given above assumes, of course, that transformations such as (5) will have the most important effects. Secondly, we note that the method presented above may be simply extended to an arbitrary number of coupled oscillations.

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